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Some Properties of Syntactic Compatibility/Incompatibility

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Abstract. In this document, we present a formal definition for syntactic compatibility. We also present definitions that can capture certain classes of syntactic incompatibility. Then, we explore certain properties of both our notion of syntactic compatibility and incompatibility.

1 Definitions

We start by building the necessary notations:

Notation 1. \( f : A \rightarrow B \) denotes a partial function from \( A \) to \( B \). For \( a \in A \) and \( b \in B \), by \( f(a) = b \), we specify both that \( f \) is defined at \( a \), and, that \( f \)'s value is \( b \) there. Furthermore, we use the notation \( f(a) = \_ \) when \( f \) is defined at \( a \) but its value there is not important to us.

Let \( \mathcal{T} \) and \( \mathcal{S} \) be the set of all types and syntactic categories, respectively. Moreover, let \( \mathfrak{A} \subseteq \mathcal{T} \) be the set of all algebraic datatypes. For an algebraic datatype \( \alpha \in \mathfrak{A} \), define \( SC(\alpha) \) to be the set of \( \alpha \)'s syntactic categories. Fix also a set \( \mathfrak{N} \) of (type constructor) names.

Definition 1. Call a partial injective function \( f : \mathfrak{A} \times \mathfrak{N} \rightarrow \mathcal{S} \) a characteristic function of \( \alpha \) (written as \( f \models \alpha \)) when \( \{ f(\alpha, n) \mid n \in \mathfrak{N} \} = SC(\alpha) \).

Note that the only algebraic datatype which \( f \) is guaranteed to be defined on is \( \alpha \), where \( f \models \alpha \). That is, whilst \( f \) might also be defined on other algebraic datatypes than \( \alpha \), one cannot infer such information from merely knowing that it is a characteristic function of \( \alpha \).

Notation 2. When \( f \) is a characteristic function, we overload the notation to \( f(\alpha) = \{ f(\alpha, n) \mid n \in \mathfrak{N} \} \) whenever \( f(\alpha, n) = \_ \) (for an arbitrary \( \alpha \)). Additionally, when \( g \models \alpha' \), write \( \mathfrak{N}_g \) for \( \{ n \in \mathfrak{N} \mid g(\alpha', n) = \_ \} \). Likewise, for \( N \subseteq \mathfrak{N}_f \), by \( f|_N \), we mean the function \( g \) such that \( g(\alpha, n) = f(\alpha, n) \) for \( n \in N \) and \( g(\alpha, n) \) is undefined otherwise.

Definition 2. Call an algebraic datatype \( \alpha_2 \) a compatible extension of an algebraic datatype \( \alpha_1 \) (written as \( \alpha_2 >_C \alpha_1 \)) when there exists a characteristic function \( f \) for \( \alpha_1 \) such that
In that case, call $f$ a witness for $\alpha_2 \succeq \alpha_1$ and write $f \models \alpha_2 \succeq \alpha_1$.

**Notation 3.** Write $\succeq_\mathcal{C}$ for the reflexive closure of $\succ$. Furthermore, call $f$ a witness for $\alpha_2 \succeq_\mathcal{C} \alpha_1$ when: either $f$ is a characteristic function of both $\alpha_1$ and $\alpha_2$, or $f$ is a witness for $\alpha_2 \succeq_\mathcal{C} \alpha_1$.

**Definition 3.** $\alpha_2$ is called $f$-incompatible with $\alpha_1$ (written as $\alpha_2 \not\succeq f \alpha_1$) when $f \models \alpha_2$ and $f(\alpha_2, n) \neq f(\alpha_1, n)$ for some $n \in \mathcal{N}_f$. Write $\alpha_2 \not\succeq f \alpha_1$ when there exists a characteristic function $f$ of $\alpha_2$ such that $\alpha_2 \not\succeq f \alpha_1$. Call $\alpha_2$ incompatible with $\alpha_1$ — written as $\alpha_2 \not\succeq \alpha_1$ — when $\alpha_2 \not\succeq f \alpha_1$ for every characteristic function $f$ of $\alpha_2$.

**Definition 4.** $\alpha_2$ always binds $\alpha_1$ (written as $\alpha_2 \succeq \alpha_1$) when, for every characteristic function $f$ of $\alpha_2$, there exists a set $N \subset \mathcal{N}_f$ such that $\{f(\alpha_1, n) \mid n \in N\} = SC(\alpha_1)$.

## 2 Properties

**Theorem 1 (Well-Definedness).** $\alpha_2 \succeq f \alpha_1$ if and only if $\alpha_1 \not\succeq f \alpha_2$.

**Proof.** Suppose that $\alpha_2 \not\succeq f \alpha_1$ and $\alpha_1 \not\succeq f \alpha_2$. By Definition 3, because $\alpha_2 \not\succeq f \alpha_1$, for every $f \models \alpha_2$, there exists $n \in \mathcal{N}_f$ such that $f(\alpha_2, n) \neq f(\alpha_1, n)$. On the other hand, because $\alpha_1 \not\succeq f \alpha_2$, by Definition 1, there exists an $f' \models \alpha_2$ such that, $f'(\alpha_2, n) = f'(\alpha_1, n)$, for every $n \in \mathcal{N}_f$. But, this is a contradiction and the result follows. \(\square\)

**Lemma 1.** Let $\alpha_1$ and $\alpha_2$ be two algebraic datatypes such that $\alpha_2 \succeq \alpha_1$. Then, $SC(\alpha_2) \supset SC(\alpha_1)$.

**Proof.** By Definition 2, $\alpha_2 \succeq_\mathcal{C} \alpha_1$ means that there exists a characteristic function $f$ for $\alpha_1$ such that:

$$f(\alpha_2) = f(\alpha_1) \quad (1)$$

$$SC(\alpha_2) \supset f(\alpha_2) \quad (2)$$

On the other hand, by Definition 1:

$$f(\alpha_1) = SC(\alpha_1) \quad (3)$$

Hence,

$$SC(\alpha_2) \supset f(\alpha_2) \supset f(\alpha_1) \supset SC(\alpha_1). \quad (\square)$$

**Lemma 2.** Let $\alpha_1$ and $\alpha_2$ be two algebraic datatypes such that $\alpha_2 \not\succeq \alpha_1$. Then, $SC(\alpha_2) \not\succeq SC(\alpha_1)$.
Proof. Assume that:
\[ SC(\alpha_2) \supset SC(\alpha_1). \]  
(4)
Let \( SC(\alpha_1) = \{s_i\}_1^k \). Pick arbitrary distinct names \( N = \{n_1, \ldots, n_k\} \), and define function \( f : \mathfrak{A} \times \mathfrak{N} \rightarrow \mathfrak{S} \) such that:
\[ \forall \alpha \in \mathfrak{A}, \forall i \in \{1, \ldots, k\}, \exists n_i \in N. f(\alpha, n_i) = s_i. \]  
(5)
By Definition 1, \( f \) is a characteristic function for \( \alpha_1 \) because \( f(\alpha_1) = SC(\alpha_1) \).
Furthermore, \( \forall n \in N. f(\alpha_1, n) = f(\alpha_2, n) \), and:
\[ f(\alpha_2) = f(\alpha_1) = SC(\alpha_1) \supset SC(\alpha_2). \]  
Hence, by Definition 2, \( \alpha_2 \succ \alpha_1 \), which is a contradiction. \( \square \)

Proposition 1. \( \succ \) is transitive.

Proof. Choose arbitrary \( \alpha_1, \alpha_2, \alpha_3 \in \mathfrak{A} \) such that \( \alpha_2 \succ \alpha_1 \) and \( \alpha_3 \succ \alpha_2 \). According to Lemma 1, \( \alpha_2 \succ \alpha_1 \) implies that \( SC(\alpha_2) \supset SC(\alpha_1) \). Similarly, \( \alpha_3 \succ \alpha_2 \) implies that \( SC(\alpha_3) \supset SC(\alpha_2) \). It follows that \( SC(\alpha_3) \supset SC(\alpha_1) \), which, by the contrapositive of Lemma 2, implies that \( \alpha_3 \succ \alpha_1 \). \( \square \)

Lemma 3. Suppose that \( f \models \alpha_3 \succ \alpha_2 \) and \( \alpha_2 \not\succ \alpha_1 \). Then, \( \alpha_3 \not\succ \alpha_1 \).

Proof. Since the result is trivial when \( f \) is a characteristic function of both \( \alpha_3 \) and \( \alpha_2 \), we only present the proof for the other case that \( f \models \alpha_3 \succ \alpha_2 \). Hence, by Definition 2, \( f \) is a characteristic function of \( \alpha_2 \) and in particular:
\[ f(\alpha_2) = SC(\alpha_2) \]  
(6)
\[ \forall n \in \mathfrak{N}_f. f(\alpha_3, n) = f(\alpha_2, n) \]  
(7)
Suppose also that
\[ \{s_i\}_1^k = SC(\alpha_3) - SC(\alpha_2). \]  
(8)
Take \( n_1, \ldots, n_k \) fresh in \( \mathfrak{N} \). Define the partial function \( g : \mathfrak{A} \times \mathfrak{N} \rightarrow \mathfrak{S} \) such that:
\[ g(\alpha, n) = \begin{cases} f(\alpha, n) & n \in \mathfrak{N}_f, \\ s_i & n = n_i \text{ for some } i \in \{1, \ldots, k\}. \end{cases} \]  
(9)
Note first that, by Definition 1, \( g \) is a characteristic function of \( \alpha_3 \) because
\[ g(\alpha_3) = \{g(\alpha_3, n) \mid n \in \mathfrak{N}\} \]  
by Notation 2
\[ = \{g(\alpha_3, n) \mid n \in \mathfrak{N}_f\} \cup \{g(\alpha_3, n) \mid n \in \{n_1, \ldots, n_k\}\} \]  
by Equation 9
\[ = \{f(\alpha_3, n) \mid n \in \mathfrak{N}_f\} \cup \{s_i\}_1^k \]  
by Equation 9
\[ = \{f(\alpha_2, n) \mid n \in \mathfrak{N}_f\} \cup \{s_i\}_1^k \]  
by Equation 7
\[ = SC(\alpha_2) \cup \{s_i\}_1^k \]  
by Equation 6
\[ = SC(\alpha_3) \]  
by Equation 8
Next, observe that, by Definition 3, \( \alpha_2 \not\succ \alpha_1 \) implies that
\[ \exists n \in \mathfrak{N}_f. f(\alpha_2, n) \neq f(\alpha_1, n) \]  
(7)
\[ \exists n \in \mathfrak{N}_f. f(\alpha_3, n) \neq f(\alpha_1, n) \]  
(9)
Hence, by Definition 3, \( \alpha_3 \not\succ \alpha_1 \), and, the result follows. \( \square \)
Lemma 4. Let $\alpha_2 \succ_{\forall} \alpha_1$ and $\alpha_2 \not\succ_{\forall} \alpha_1$. Then, $\alpha_2 \not\succ_{f} \alpha_1$ whenever $f$ is a characteristic function of $\alpha_2$.

Proof. Suppose that
\[ \alpha_2 \not\succ_{\forall} \alpha_1 \]  
\[ \alpha_2 \succ_{\forall} \alpha_1 \]  
and that the lemma is wrong. That is, $f \models \alpha_2$ and $\forall n \in \mathcal{R}_f, f(\alpha_1, n) = f(\alpha_2, n)$. Given that $f \models \alpha_2$, by Definition 4, we conclude from (11) that
\[ \exists N \subset \mathcal{R}_f, \{ f(\alpha_1, n) \mid n \in N \} = SC(\alpha_1) \]
(12)
Define $g : \mathfrak{A} \times \mathcal{R} \to \mathcal{G}$ such that for all $n \in N$
\[ g(\alpha, n) = f(\alpha, n) \]  
(13)
Note first that
\[ g(\alpha_1) = \{ g(\alpha_1, n) \mid n \in N \} \overset{\text{(13)}}{=} \{ f(\alpha_1, n) \mid n \in N \} \overset{\text{(12)}}{=} SC(\alpha_1) \]
which, by Definition 1, means that $g \models \alpha_1$. Next, observe that
\[ N \subset \mathcal{R}_f \Rightarrow \{ f(\alpha_2, n) \mid n \in N \} \subset \{ f(\alpha_2, n) \mid n \in \mathcal{R}_f \} \Rightarrow \]
by Definition 13
\[ \{ g(\alpha_2, n) \mid n \in N \} \subset \{ f(\alpha_2, n) \mid n \in \mathcal{R}_f \} \Rightarrow \]
by Notation 2
\[ g(\alpha_2) \subset SC(\alpha_2) \]
(15)
which, by Definition 4, means that $g \models \alpha_2 \succ_{\forall} \alpha_1$, which contradicts with (10). Hence, the result follows.

Proposition 2. Suppose that $\alpha_3 \geq_{\forall} \alpha_2$, $\alpha_2 \not\succ_{\forall} \alpha_1$, and $\alpha_2 \succ_{\forall} \alpha_1$. Then, $\alpha_3 \not\succ_{\forall} \alpha_1$.

Proof. By Definition 2, $\alpha_3 \geq_{\forall} \alpha_2$ means that there exists a characteristic function $f$ of $\alpha_2$ such that
\[ f \models \alpha_3 \geq_{\forall} \alpha_2 \]  
(14)
Given that $f \models \alpha_2$, because $\alpha_2 \not\succ_{\forall} \alpha_1$ and $\alpha_2 \succ_{\forall} \alpha_1$, by Lemma 4
\[ \alpha_2 \not\succ_{f} \alpha_1 \]  
(15)
The result follows from (14) and (15) using Lemma 3.

Lemma 5. Suppose that $\alpha_2 \succ_{\forall} \alpha_1$ and $f_2 \models \alpha_2$. Then, there exists an $f_1 \models \alpha_1$ such that $f_1 = f_2|_N$ for some $N \subset \mathcal{R}_{f_2}$.

Proof. By Definition 4, because $\alpha_2 \succ_{\forall} \alpha_1$ and $f_2 \models \alpha_2$, we know that there exists a set $N' \subset \mathcal{R}_{f_2}$ such that $\{ f_2(\alpha_1, n) \mid n \in N' \} = SC(\alpha_1)$. Choose $N \subseteq N'$ such that $f_1 = f_2|_N$ is injective. By Definition 1, $f_1 \models \alpha_1$. □
Corollary 1. Suppose that $\alpha_2 \succ \succ \alpha_1$ and $\alpha_2 \succ \succ \alpha_1$. Then, for every $f_2 \models \alpha_2$, there exists an $f_1 \models \alpha_2 \succ \succ \alpha_1$ such that $f_1 = f_2|_N$ for some $N \subset \mathcal{R}|f_2$.

Proof. Suppose that $\alpha_2 \succ \succ \alpha_1$ and $\alpha_2 \succ \succ \alpha_1$. Fix an arbitrary $f_2 \models \alpha_2$. Given that, $\alpha_2 \succ \succ \alpha_1$, Lemma 5 already implies that there exists an $f_1 \models \alpha_1$ such that $f_1 = f_2|_N$ for some $N \subset \mathcal{R}|f_2$. It remains to prove that $f_1 \models \alpha_2 \succ \succ \alpha_1$. To that end, notice first that $f_1 = f_2|_N$ means that

$$\forall n \in N. \ f_1(\alpha_2, n) = f_1(\alpha_1, n) \quad (16)$$

Secondly, by Definition 1, $f_1 \models \alpha_1$ entails $f_1(\alpha_1) = SC(\alpha_1)$. By Equation 16, it follows that $f_1(\alpha_2) = SC(\alpha_1)$ too. (Note that $f_1$ is only defined when $n \in N$.) Moreover, by Lemma 1, $\alpha_2 \succ \alpha_1$ implies $SC(\alpha_2) \supset SC(\alpha_1)$. Thus,

$$SC(\alpha_2) \supset f_1(\alpha_2) \quad (17)$$

By Definition 2, the result follows from Equations 16 and 17. \qed

Proposition 3. Suppose that $\alpha_3 \succ \succ \alpha_2$, $\alpha_3 \succ \succ \alpha_2$, $\alpha_2 \not\succ \alpha_1$, and $\alpha_2 \succ \succ \alpha_1$. Then, $\alpha_3 \not\succ \alpha_1$.

Proof. The result is trivial when $\alpha_3 = \alpha_2$. In order to prove it for the other case, choose an arbitrary $f_3 \models \alpha_3$. Because $\alpha_3 \succ \succ \alpha_2$ and $\alpha_3 \succ \succ \alpha_2$, according to Corollary 1, there exists an $f_2 \models \alpha_2 \succ \succ \alpha_2$ such that

$$f_2 = f_3|_{N_2} \quad (18)$$

for some $N_2 \subset \mathcal{R}|f_3$. Furthermore, according to Lemma 5, one concludes from $\alpha_2 \succ \succ \alpha_1$ that there exists an $f_1 \models \alpha_1$ such that

$$f_1 = f_2|_{N_1} \quad (19)$$

for some $N_1 \subset N_2$. The result follows if we can show that $\alpha_3 \not\succ f_3 \alpha_4$. Suppose otherwise. That is,

$$\forall n \in N_1. \ f_3(\alpha_3, n) = f_3(\alpha_1, n) \Rightarrow \quad \text{by Equation 18}$$

$$\forall n \in N_1. \ f_2(\alpha_3, n) = f_2(\alpha_1, n) \Rightarrow \ f_2 \models \alpha_3 \succ \alpha_2$$

$$\forall n \in N_1. \ f_2(\alpha_2, n) = f_2(\alpha_1, n)$$

which, because of Equation 19, implies that

$$\forall n \in N_1. \ f_1(\alpha_2, n) = f_1(\alpha_1, n) \quad (20)$$

On the other hand,

$$f_1(\alpha_2) = \{ f_1(\alpha_2, n) \mid n \in N_1 \} = \{ f_2(\alpha_2, n) \mid n \in N_1 \} \subset \{ f_2(\alpha_2, n) \mid n \in N_2 \}$$

That is, $f_1(\alpha_2) \subset f_2(\alpha_2)$. Besides, because $f_2 \models \alpha_2$, it follows by Definition 1, $f_2(\alpha_2) = SC(\alpha_2)$. Thus,

$$f_1(\alpha_2) \subset SC(\alpha_2) \quad (21)$$

Remembering that $f_1 \models \alpha_1$, by Definition 2, one concludes $\alpha_2 \succ \succ \alpha_1$ from Equations 20 and 21 – which is a contradiction. \qed