

# Uncertainty

## Chapter 13

Slides by Hwee Tou Ng (Singapore)

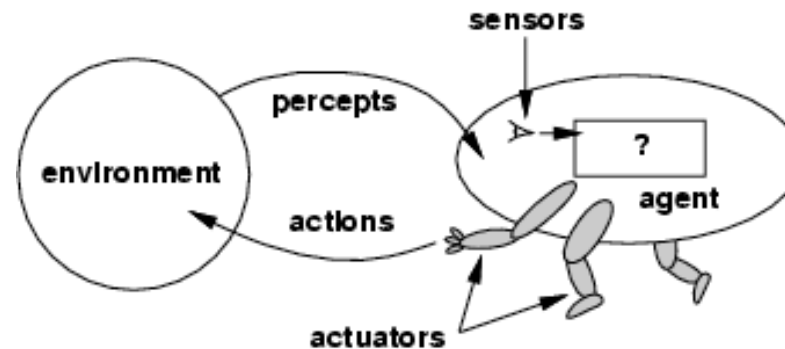
# Outline

- Agents
- Uncertainty
- Probability
- Syntax and Semantics
- Inference
- Independence and Bayes' Rule

# Agents

- An **agent** is anything that can be viewed as **perceiving** its **environment** through **sensors** and **acting** upon that environment through **actuators**
- Human agent: eyes, ears, and other organs for sensors; hands, legs, mouth, and other body parts for actuators
- Robotic agent: cameras and infrared range finders for sensors;
- various motors for actuators

# Agents and environments



- The **agent function** maps from percept histories to actions:

$$[f: P^* \rightarrow \mathcal{A}]$$

- The **agent program** runs on the physical **architecture** to produce  $f$
- agent = architecture + program  
architecture: PC, robotic car, ...

# Uncertainty

Let action  $A_t$  = leave for airport  $t$  minutes before flight  
Will  $A_t$  get me there on time?

Problems:

1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports)
3. uncertainty in action outcomes (flat tire, etc.)
4. immense complexity of modeling and predicting traffic

Hence a purely logical approach either

1. risks falsehood: " $A_{25}$  will get me there on time", or
2. leads to conclusions that are too weak for decision making:

" $A_{25}$  will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

( $A_{1440}$  might reasonably be said to get me there on time but I'd have to stay overnight in the airport ...)

# Methods for handling uncertainty

- **Logic:**
  - Assume my car does not have a flat tire
  - Assume  $A_{25}$  works unless contradicted by evidence
- Issues: What assumptions are reasonable? How to handle contradiction?
- **Rules with fudge factors (belief in the rule):**
  - $A_{25} \mid\rightarrow_{0.3}$  get there on time
  - $Sprinkler \mid\rightarrow_{0.99} WetGrass$
  - $WetGrass \mid\rightarrow_{0.7} Rain$
- Issues: Problems with combination, e.g., *Sprinkler causes Rain??*
- **Probability**
  - Model agent's degree of belief
  - Given the available evidence,
  - $A_{25}$  will get me there on time with probability 0.04

# Probability

Probabilistic assertions **summarize** effects of

- **laziness**: failure to enumerate exceptions, qualifications, etc.
- **ignorance**: lack of relevant facts, initial conditions, etc.

**Subjective** probability:

- Probabilities relate propositions to agent's own state of knowledge

$$\text{e.g., } P(A_{25} \mid \text{no reported accidents}) = 0.06$$

These are **not** assertions about the world

Probabilities of propositions change with new evidence:

$$\text{e.g., } P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$$

# Making decisions under uncertainty

Suppose I believe the following:

$$P(A_{25} \text{ gets me there on time} \mid \dots) = 0.04$$

$$P(A_{90} \text{ gets me there on time} \mid \dots) = 0.70$$

$$P(A_{120} \text{ gets me there on time} \mid \dots) = 0.95$$

$$P(A_{1440} \text{ gets me there on time} \mid \dots) = 0.9999$$

- Which action to choose?

Depends on my **preferences** for missing flight vs. time spent waiting, etc.

- **Utility theory** is used to represent and infer preferences
- **Decision theory** = probability theory + utility theory

# Syntax

- Basic element: **random variable**
- Similar to propositional logic: possible worlds defined by assignment of values to random variables.
- **Boolean** random variables  
e.g., *Cavity* (do I have a cavity?). Domain is  $\langle \text{true}, \text{false} \rangle$
- **Discrete** random variables  
e.g., *Weather* is one of  $\langle \text{sunny}, \text{rainy}, \text{cloudy}, \text{snow} \rangle$
- Domain values must be exhaustive and mutually exclusive
- Elementary proposition constructed by assignment of a value to a random variable:  
e.g., *Weather* = *sunny*, *Cavity* = *false* (abbreviated as  $\neg \text{cavity}$ )
- Complex propositions formed from elementary propositions and standard logical connectives e.g., *Weather* = *sunny*  $\vee$  *Cavity* = *false*

# Syntax

- **Atomic event:** A **complete** specification of the state of the world about which the agent is uncertain

E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:

*Cavity* = *false*  $\wedge$  *Toothache* = *false*

*Cavity* = *false*  $\wedge$  *Toothache* = *true*

*Cavity* = *true*  $\wedge$  *Toothache* = *false*

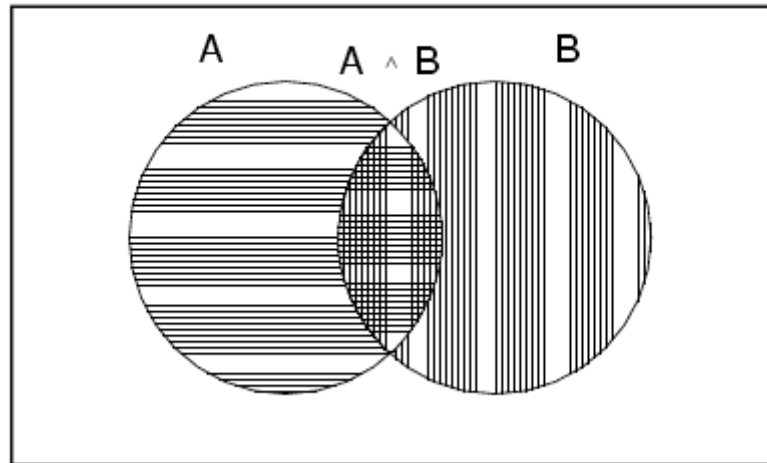
*Cavity* = *true*  $\wedge$  *Toothache* = *true*

- Atomic events are mutually exclusive and exhaustive

# Axioms of probability

- For any propositions  $A$ ,  $B$ 
  - $0 \leq P(A) \leq 1$
  - $P(\text{true}) = 1$  and  $P(\text{false}) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

True



# Example World

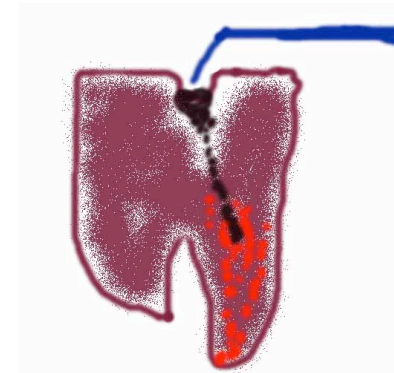
**Example:** *Dentist problem* with four events:

*Toothache* (I have a toothache)

*Cavity* (I have a cavity)

*Catch* (steel probe catches in my tooth)

*Weather* ( *sunny,rainy,cloudy,snow* )



# Prior probability

- **Prior or unconditional probabilities** of propositions  
e.g.,  $P(\text{Cavity} = \text{true}) = 0.1$  and  $P(\text{Weather} = \text{sunny}) = 0.72$  correspond to belief prior to arrival of any (new) evidence
- **Probability distribution** gives values for all possible assignments:  
 $\mathbf{P}(\text{Weather}) = \langle 0.72, 0.1, 0.08, 0.1 \rangle$  (**normalized**, i.e., sums to 1)

# Full joint probability distribution

- **Joint probability distribution** for a set of random variables gives the probability of every atomic event on those random variables

$P(\text{Weather}, \text{Cavity})$  = a  $4 \times 2$  matrix of values:

<i>Weather</i> =	sunny	rainy	cloudy	snow
<i>Cavity</i> = true	0.144	0.02	0.016	0.02
<i>Cavity</i> = false	0.576	0.08	0.064	0.08

- Full joint probability distribution: all random variables involved
  - $P(\text{Toothache}, \text{Catch}, \text{Cavity}, \text{Weather})$
- Every question about a domain can be answered by the full joint distribution

# Conditional probability

- Conditional or posterior probabilities  
e.g.,  $P(\text{cavity} \mid \text{toothache}) = 0.8$   
i.e., given that *toothache* is all I know
- (Notation for conditional distributions:  
 $\mathbf{P}(\text{Cavity} \mid \text{Toothache}) = 2\text{-element vector of } 2\text{-element vectors}$ )
- If we know more, e.g., *cavity* is also given, then we have  
 $P(\text{cavity} \mid \text{toothache}, \text{cavity}) = 1$
- New evidence may be irrelevant, allowing simplification, e.g.,  
 $P(\text{cavity} \mid \text{toothache}, \text{sunny}) = P(\text{cavity} \mid \text{toothache}) = 0.8$
- This kind of inference, sanctioned by domain knowledge, is crucial

# Conditional probability

- Definition of conditional probability (in terms of uncond. prob.):  
 $P(a | b) = P(a \wedge b) / P(b)$  if  $P(b) > 0$
- **Product rule** gives an alternative formulation ( $\wedge$  is commutative):  
 $P(a \wedge b) = P(a | b) P(b) = P(b | a) P(a)$
- A general version holds for whole distributions, e.g.,  
 $P(\textit{Weather}, \textit{Cavity}) = P(\textit{Weather} | \textit{Cavity}) P(\textit{Cavity})$   
View as a set of  $4 \times 2$  equations, **not** matrix mult.  
(1,1)  $P(\textit{Weather}=\textit{sunny} | \textit{Cavity}=\textit{true}) P(\textit{Cavity}=\textit{true})$   
(1,2)  $P(\textit{Weather}=\textit{sunny} | \textit{Cavity}=\textit{false}) P(\textit{Cavity}=\textit{false}), \dots$
- **Chain rule** is derived by successive application of product rule:  
$$\begin{aligned} P(X_1, \dots, X_n) &= P(X_1, \dots, X_{n-1}) P(X_n | X_1, \dots, X_{n-1}) \\ &= P(X_1, \dots, X_{n-2}) P(X_{n-1} | X_1, \dots, X_{n-2}) P(X_n | X_1, \dots, X_{n-1}) \\ &= \dots \\ &= \prod_{i=1}^n P(X_i | X_1, \dots, X_{i-1}) \end{aligned}$$

# Inference by enumeration

- Start with the joint probability distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

- For any proposition  $\varphi$ , sum the atomic events where it is true:  $P(\varphi) = \sum_{\omega:\omega \models \varphi} P(\omega)$

# Inference by enumeration

- Start with the joint probability distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

- For any proposition  $\phi$ , sum the atomic events where it is true:  $P(\phi) = \sum_{\omega:\omega \models \phi} P(\omega)$
- $P(\text{toothache}) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2$
- Unconditional or **marginal probability** of toothache
- Process is called marginalization or summing out

# Inference by enumeration

- Start with the joint probability distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	<b>.072</b>	<b>.008</b>
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	<b>.144</b>	<b>.576</b>

- For any proposition  $\varphi$ , sum the atomic events where it is true:  $P(\varphi) = \sum_{\omega: \omega \models \varphi} P(\omega)$
- $P(\text{cavity} \vee \text{toothache}) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$

$$(P(\text{cavity} \vee \text{toothache}) = P(\text{cavity}) + P(\text{toothache}) - P(\text{cavity} \wedge \text{toothache}))$$

# Inference by enumeration

- Start with the joint probability distribution:

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	.072	.008
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	.144	.576

- Can also compute conditional probabilities:

$$\begin{aligned} P(\neg \text{cavity} \mid \text{toothache}) &= \frac{P(\neg \text{cavity} \wedge \text{toothache})}{P(\text{toothache})} \\ &\stackrel{\text{Product rule}}{=} \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} \\ &= 0.4 \end{aligned}$$

# Conditioning Rule

- $P(\mathbf{Y}) = \sum_{\mathbf{z}} P(\mathbf{Y}|\mathbf{z})P(\mathbf{z})$

# Normalization

	<i>toothache</i>		$\neg$ <i>toothache</i>	
	<i>catch</i>	$\neg$ <i>catch</i>	<i>catch</i>	$\neg$ <i>catch</i>
<i>cavity</i>	<b>.108</b>	<b>.012</b>	.072	.008
$\neg$ <i>cavity</i>	<b>.016</b>	<b>.064</b>	.144	.576

- Denominator  $\mathbf{P}(\mathbf{z})$  (or  $\mathbf{P}(\text{toothache})$  in the example before) can be viewed as a **normalization constant**  $\alpha$

$$\begin{aligned}\mathbf{P}(\text{Cavity} \mid \text{toothache}) &= \alpha \mathbf{P}(\text{Cavity}, \text{toothache}) \\ &= \alpha [\mathbf{P}(\text{Cavity}, \text{toothache}, \text{catch}) + \mathbf{P}(\text{Cavity}, \text{toothache}, \neg \text{catch})] \\ &= \alpha [ \langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle ] \\ &= \alpha \langle 0.12, 0.08 \rangle = \langle 0.6, 0.4 \rangle\end{aligned}$$

General idea: compute distribution on query variable by fixing **evidence variables** (Toothache) and summing over **hidden variables** (Catch)

# Inference by enumeration, contd.

Typically, we are interested in  
the posterior joint distribution of the **query variables**  $\mathbf{Y}$   
given specific values  $\mathbf{e}$  for the **evidence variables**  $\mathbf{E}$

Let the **hidden variables** be  $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$

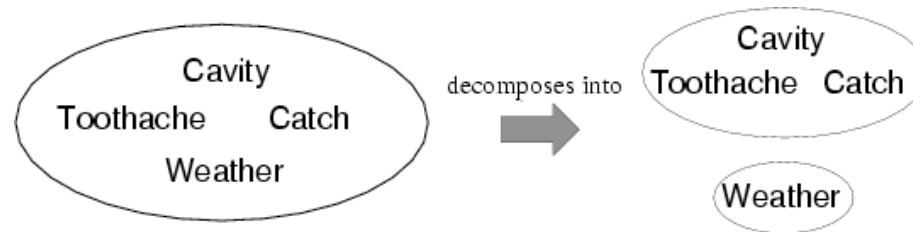
Then the required summation of joint entries is done by summing out the hidden variables:

$$P(\mathbf{Y} \mid \mathbf{E} = \mathbf{e}) = \alpha P(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} P(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h})$$

- The terms in the summation are joint entries because  $\mathbf{Y}$ ,  $\mathbf{E}$  and  $\mathbf{H}$  together exhaust the set of random variables
- Obvious problems:
  1. Worst-case time complexity  $O(d^n)$  where  $d$  is the largest arity and  $n$  denotes the number of random variables
  2. Space complexity  $O(d^n)$  to store the joint distribution
  3. How to find the numbers for  $O(d^n)$  entries?

# Independence

- $A$  and  $B$  are independent iff  
 $\mathbf{P}(A|B) = \mathbf{P}(A)$  or  $\mathbf{P}(B|A) = \mathbf{P}(B)$  or  $\mathbf{P}(A, B) = \mathbf{P}(A) \mathbf{P}(B)$



$$\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}, \textit{Weather}) \\ = \mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}) \mathbf{P}(\textit{Weather})$$

- 32 entries reduced to 12;
- for  $n$  independent biased coins,  $O(2^n) \rightarrow O(n)$
- Absolute independence powerful but rare
- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

# Conditional independence

- $\mathbf{P}(\textit{Toothache}, \textit{Cavity}, \textit{Catch})$  has  $2^3 - 1 = 7$  independent entries
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:  
(1)  $\mathbf{P}(\textit{catch} \mid \textit{toothache}, \textit{cavity}) = \mathbf{P}(\textit{catch} \mid \textit{cavity})$
- The same independence holds if I haven't got a cavity:  
(2)  $\mathbf{P}(\textit{catch} \mid \textit{toothache}, \neg \textit{cavity}) = \mathbf{P}(\textit{catch} \mid \neg \textit{cavity})$
- *Catch* is **conditionally independent** of *Toothache* given *Cavity*:  
 $\mathbf{P}(\textit{Catch} \mid \textit{Toothache}, \textit{Cavity}) = \mathbf{P}(\textit{Catch} \mid \textit{Cavity})$
- Equivalent statements:  
 $\mathbf{P}(\textit{Toothache} \mid \textit{Catch}, \textit{Cavity}) = \mathbf{P}(\textit{Toothache} \mid \textit{Cavity})$   
 $\mathbf{P}(\textit{Toothache}, \textit{Catch} \mid \textit{Cavity}) = \mathbf{P}(\textit{Toothache} \mid \textit{Cavity}) \mathbf{P}(\textit{Catch} \mid \textit{Cavity})$

# Conditional independence contd.

- Write out full joint distribution using chain rule:

$\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})$

$$= \mathbf{P}(\textit{Toothache} \mid \textit{Catch}, \textit{Cavity}) \mathbf{P}(\textit{Catch}, \textit{Cavity})$$

$$= \mathbf{P}(\textit{Toothache} \mid \textit{Catch}, \textit{Cavity}) \mathbf{P}(\textit{Catch} \mid \textit{Cavity}) \mathbf{P}(\textit{Cavity})$$

$$= \mathbf{P}(\textit{Toothache} \mid \textit{Cavity}) \mathbf{P}(\textit{Catch} \mid \textit{Cavity}) \mathbf{P}(\textit{Cavity})$$

I.e.,  $2 + 2 + 1 = 5$  independent numbers

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in  $n$  to linear in  $n$ .
- Conditional independence is our most basic and robust form of knowledge about uncertain environments.

# Bayes' Rule

Product rule  $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$$\Rightarrow \text{Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

or in distribution form

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)} = \alpha P(X|Y)P(Y)$$

Useful for assessing **diagnostic** probability from **causal** probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

E.g., let  $M$  be meningitis,  $S$  be stiff neck:

$$P(m|s) = \frac{P(s|m)P(m)}{P(s)} = \frac{0.8 \times 0.0001}{0.1} = 0.0008$$

Note: posterior probability of meningitis still very small!

# Bayes' Rule (2)

$$\begin{aligned} & \mathbf{P}(Cavity|toothache \wedge catch) \\ &= \alpha \mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity) \\ &= \alpha \mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity) \end{aligned}$$

This is an example of a **naive Bayes** model:

$$\mathbf{P}(Cause, Effect_1, \dots, Effect_n) = \mathbf{P}(Cause) \prod_i \mathbf{P}(Effect_i|Cause)$$



Total number of parameters is **linear** in  $n$